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Wick's theorem for q-deformed boson operators

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Abstract

In this paper, combinatorial aspects of normal ordering of arbitrary words in the creation and annihilation operators of the q-deformed boson are discussed. In particular, it is shown how, by introducing appropriate q-weights for the associated 'Feynman diagrams', the normally ordered form of a general expression in the creation and annihilation operators can be written as a sum over all q-weighted Feynman diagrams, representing Wick's theorem in the present context.

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1. Introduction

Since the seminal work of Katriel [1], the combinatorial aspects of normal ordering of arbitrary words in the creation and annihilation operators $b^\dagger$ and b of a single-mode boson have been studied intensively and many generalizations have also been considered; see, e.g., [2–18] and the references given therein. For an important discussion and references to the earlier literature on normal ordering of noncommuting operators, see [19]. For the creation and annihilation operators $f^\dagger$ and f of a single-mode fermion, respectively, the analogous combinatorial problem does not exist due to the nilpotency of the operators, i.e. $(f^\dagger)^2 = 0 = f^2$. However, if one considers instead of a single-mode fermion a multi-mode fermion (i.e. several sets of operators $f_i, f_i^\dagger$), then interesting combinatorial connections to rook numbers exist (note that the general normal ordering problem of a single-mode boson has also straightforward connections to rook numbers [14]). This was noted by Navon [20] even before Katriel demonstrated that normal ordering powers of the bosonic number operator, i.e. $(b^\dagger b)^n$, involve the Stirling numbers of the second kind [1]. Very recently, combinatorial aspects of multi-mode boson operators (where the different modes interact due to a nontrivial commutation relation) have also been investigated [18]. Starting with the paper of Katriel and Kibler [21], the combinatorial aspects of normal ordering arbitrary words in the creation and annihilation operators $c^\dagger$ and c of a single-mode q-boson having the commutation relations

$$[c, c^\dagger]_q \equiv cc^\dagger - qc^\dagger c = 1, \qquad [c, c] = 0, \qquad [c^\dagger, c^\dagger] = 0 \tag{1}$$

have also been considered [8, 14, 21–23]. Recall that *normal ordering* is a functional representation of operator functions in which all the creation operators stand to the left of the annihilation operators. Let an arbitrary operator function $F(c, c^\dagger)$ be given; a function $F(c, c^\dagger)$ can be seen as a word on the alphabet $\{c, c^\dagger\}$. We denote by $\mathcal{N}_q[F(c, c^\dagger)]$ the normal ordering of the function $F(c, c^\dagger)$. Clearly, the operation of normal ordering $\mathcal{N}_q$ is a linear map, that is, $\mathcal{N}_q(F(c, c^\dagger) + G(c, c^\dagger)) = \mathcal{N}_q(F(c, c^\dagger)) + \mathcal{N}_q(G(c, c^\dagger))$ for any two operator functions $F(c, c^\dagger)$ and $G(c, c^\dagger)$ respectively. Using the commutation relations (1), it is clear that their normally ordered form $\mathcal{N}_q[F(c, c^\dagger)] = F(c, c^\dagger)$ can be written as

$$\mathcal{N}_q[F(c, c^\dagger)] = F(c, c^\dagger) = \sum_{k,l} C_{k,l}(q)(c^\dagger)^k c^l \tag{2}$$

for some coefficients $C_{k,l}(q)$, and the main task consists of determining the coefficients as explicitly as possible.

For example, Katriel and Kibler showed [21] that normal ordering the powers of $c^\dagger c$ involves the q-deformed Stirling numbers of the second kind $S_q(n, k)$ in the version of Milne [24], i.e.

$$\mathcal{N}_q[(c^\dagger c)^n] = \sum_{k=0}^{n} S_q(n, k)(c^\dagger)^k c^k. \tag{3}$$

Note that when introducing Fock space representations one has $c^\dagger c = [N_c]$, where we have denoted by N_c the associated number operator and by $[x]_q = \frac{1-q^x}{1-q} = 1 + q + \cdots + q^{x-1}$ the q-deformed version of x (where x is a number or an operator). To describe the normally ordered form of an arbitrary expression is, therefore, an interesting problem. Varvak has shown [14] that the general coefficients can be interpreted as q-rook numbers. We will describe in the present paper a different approach associated with 'q-weighted Feynman diagrams'. Note that in [25–27], very similar results have been shown in slightly different situations.

The structure of the paper is as follows: In section 2 we introduce 'Feynman diagrams' associated with our problem (following closely the terminology of [27]) and introduce q-weights for these Feynman diagrams. In section 3 we state and proof Wick's theorem adapted to the present situation, i.e. describe the coefficients of (2) in terms of q-weighted Feynman diagrams. In section 4, some examples and consequences are discussed.

2. Feynman diagrams and associated *q*-weights

Recall that in the undeformed case (i.e. $q = 1$) one has

$$\mathcal{N}[F(c, c^\dagger)] = F(c, c^\dagger) = \sum_{\pi \in \mathcal{C}(F(c,c^\dagger))} : \pi :, \tag{4}$$

where we have denoted by $\mathcal{C}(F(c, c^\dagger))$ the multiset of contractions of the word $F(c, c^\dagger)$, and the double dot operation changes the order of the operators such that all creation operators precede the annihilation operators [17]. In comparison to [17, 18] we now slightly switch the terminology to match that of [27] (and also [25, 26, 28]) which we will follow closely. Let $S = \{s_1, \ldots, s_n\}$ be a finite linearly ordered set consisting of two types of elements, i.e. there exists a 'type-map' $\tau : S \to \{\mathscr{A}, \mathscr{C}\}$ which associates with each letter s_i its type, i.e. $\tau(s_i) \in \{\mathscr{A}, \mathscr{C}\}$. We call elements s_i with $\tau(s_i) = \mathscr{A}$ 'annihilators' and elements s_j with

$\tau(s_j) = \mathscr{C}$ 'creators'. We also denote by S^+ (respectively S^-) the set of j with $\tau(s_j) = \mathscr{C}$ (respectively i with $\tau(s_i) = \mathscr{A}$). A *Feynman diagram* γ on S is a partition of S into one- and two-element sets, where the two-element sets have the special property that the two elements are of a different type (i.e. contain exactly one creator and one annihilator) and where the element of type $\mathscr{C}$ is the one with a larger index. We also regard γ as a set of ordered pairs $\{(i_1, j_1), \ldots, (i_p, j_p)\}$ with $i_k < j_k$ and $i_k \neq i_l$, $j_k \neq j_l$ and $s_{i_k} \in S^-$, $s_{j_k} \in S^+$. We also assume with this notation that $i_1 < i_2 < \cdots < i_p$.

Remark 2.1. Before continuing let us draw the connection to the terminology used in [17, 18]. In our concrete model, the set S is given by the word $F(c, c^\dagger)$, the two types are given by $\mathscr{C} = c^\dagger$ and $\mathscr{A} = c$, respectively, and a Feynman diagram γ corresponds precisely to a contraction. In fact, the two-element sets (i_k, j_k) correspond to the edges of the contraction connecting a creator $c^\dagger$ with a preceding annihilator c. Thus, the Feynman diagram $\gamma = \{(i_1, j_1), \ldots, (i_p, j_p)\}$ corresponds to a contraction of degree p.

Let us introduce some further terminology following [27]. Given S, we call the elements of S *vertices*. A Feynman diagram with representation $\gamma = \{(i_1, j_1), \ldots, (i_p, j_p)\}$ is said to have degree p and the two-element sets (i_k, j_k) are called *edges*. We denote the set of all Feynman diagrams on S by $\mathscr{F}(S)$ and the set of Feynman diagrams of degree p by $\mathscr{F}_p(S)$. If $|S| = n$, then $\mathscr{F}_p(S) = 0$ for $p > \frac{n}{2}$. Given a Feynman diagram $\gamma \in \mathscr{F}_p(S)$ with $2\mathrm{p} \leqslant n$, there will be $n - 2\mathrm{p}$ unpaired indices in γ to which we refer as *singletons* (in the terminology of [17, 18], these are the vertices of degree 0). The set of singletons of γ will be denoted by $\mathscr{S}(\gamma)$. Let us now introduce the double dot operation for a Feynman diagram γ on S. Intuitively, it means that we omit all vertices contained in the two-element sets of γ and order the remaining singletons in such a fashion that all creators precede the annihilators. More formally, let $\gamma \in \mathscr{F}_p(S)$ and assume that γ has r singletons of type $\mathscr{C}$ (respectively s singletons of type $\mathscr{A}$ with $s = n - 2\mathrm{p} - r$). Then we define $: \gamma : = \mathscr{C}^r \mathscr{A}^s$. Using this terminology, we can write the undeformed case (4) as

$$\mathcal{N}[F(c, c^\dagger)] = F(c, c^\dagger) = \sum_{\gamma \in \mathscr{F}(F(c,c^\dagger))} : \gamma :, \tag{5}$$

for any word $F(c, c^\dagger)$.

We now introduce a q-weight for Feynman diagrams such that we can write the normally ordered form of words $F(c, c^\dagger)$ in the q-boson operators in a form analogous to (5). For this we have to introduce some more terminology following [27]. Let $\gamma = \{(i_1, j_1), \ldots, (i_p, j_p)\}$ be a Feynman diagram. We say that a pair (i_k, j_k) is a *left crossing* for (i_m, j_m) if $i_k < i_m < j_k < j_m$, and we define $c_l(i, j)$ to be the number of such left crossings for (i, j). We define $c(\gamma) := \sum_{(i,j)\in\gamma} c_l(i, j)$ as the *crossing number* of γ (Biane calls it a *restricted crossing number* [28]); it counts the intersections in the corresponding graph (the *linear representation* of γ). We also need to count *degenerate crossings*. These are the triples $i < k < j$, where k is not paired (i.e. a singleton), and $(i, j) \in \gamma$. Letting $d(i, j)$ be the number of such unpaired k for the edge (i, j), then $d(\gamma) := \sum_{(i,j)\in\gamma} d(i, j)$ counts the number of such triples in γ. The *total crossing number* of γ is defined by

$$tc(\gamma) := c(\gamma) + d(\gamma).$$

It accounts for the 'interaction' between edges (the crossings) and the 'interaction' between singletons and edges (covering of singletons by edges). We now need in addition a measure which accounts for the 'interaction' between singletons. For a singleton s_k of γ, we define its *length* (to the right) $l_r(s_k)$ as follows: If the singleton s_k is of type $\mathscr{C}$, then $l_r(s_k) = 0$; if the singleton s_k is of type $\mathscr{A}$, then $l_r(s_k)$ is given by the number of singletons of type $\mathscr{C}$ to the

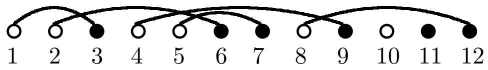

Figure 1. The linear representation of the Feynman diagram γ.

right of s_k. The *length* of γ is defined to be the sum of the lengths of all its singletons, i.e. $l(\gamma) := \sum_{s\in\mathscr{S}(\gamma)} l_r(s)$. After these lengthy preparations, we can now define the *q-weight* of a Feynman diagram γ to be

$$\mathscr{W}_q(\gamma) := q^{tc(\gamma)+l(\gamma)}. \tag{6}$$

Example 2.2. Let $F(c, c^\dagger) = ccc^\dagger ccc^\dagger c^\dagger cc^\dagger cc^\dagger c^\dagger$ and consider the Feynman diagram $\gamma = \{(1, 3), (2, 6), (4, 9), (5, 7), (8, 12)\}$ of degree 5. In its linear representation in figure 1 the vertices of type $\mathscr{A} = c$ are depicted by an empty circle, while the vertices of type $\mathscr{C} = c^\dagger$ are depicted by a black circle. The crossing number of γ is given by $c(\gamma) = 4$ and there are two degenerate crossings, i.e. $d(\gamma) = 2$, yielding the total crossing number $tc(\gamma) = 6$. There is only one singleton of type $\mathscr{A} = c$ having length $l_r(s_{10}) = 1$, yielding the length $l(\gamma) = 1$. Thus, the q-weight of γ is given by $\mathscr{W}_q(\gamma) = q^{6+1} = q^7$.

3. Wick's theorem for the *q*-deformed boson

We can now state the generalization of (5) to the q-deformed case.

Theorem 3.1. *Let $F(c, c^\dagger)$ be an operator function of the annihilation and creation operators of the q-boson (1). Then the normally ordered form $\mathcal{N}_q[F(c, c^\dagger)]$ can be described with q-weighted Feynman diagrams and the double dot operation as follows:*

$$\mathcal{N}_q[F(c, c^\dagger)] = F(c, c^\dagger) = \sum_{\gamma\in\mathscr{F}(F(c,c^\dagger))} \mathscr{W}_q(\gamma) : \gamma : . \tag{7}$$

Proof. We show this by induction in the length of the words $F(c, c^\dagger)$. Thus, assume that the relation holds for all words of length less than or equal to n. A word $G(c, c^\dagger)$ of length $n + 1$ can be written either as (I) $c^\dagger F(c, c^\dagger)$ or as (II) $cF(c, c^\dagger)$. Let us start with case (I). From the definitions, it follows that

$$\mathcal{N}_q[c^\dagger F(c, c^\dagger)] = c^\dagger \mathcal{N}_q[F(c, c^\dagger)] = \sum_{\gamma\in\mathscr{F}(F(c,c^\dagger))} \mathscr{W}_q(\gamma)c^\dagger : \gamma : = \sum_{\gamma\in\mathscr{F}(c^\dagger F(c,c^\dagger))} \mathscr{W}_q(\gamma) : \gamma :,$$

where we have used in the last equation the fact that there is a bijection between $\mathscr{F}(F(c, c^\dagger))$ and $\mathscr{F}(c^\dagger F(c, c^\dagger))$ such that we can identify $\gamma \in \mathscr{F}(F(c, c^\dagger))$ with the corresponding $\gamma \in \mathscr{F}(c^\dagger F(c, c^\dagger))$ having the same weight. This shows case (I). Let us turn to case (II). By the induction hypothesis, we assume that $\mathcal{N}_q[F(c, c^\dagger)] = \sum_{\gamma\in\mathscr{F}(F(c,c^\dagger))} \mathscr{W}_q(\gamma) : \gamma :$. Thus, from the definition of the q-normal ordering we can state

$$\mathcal{N}_q[cF(c, c^\dagger)] = \mathcal{N}_q[c\mathcal{N}_q[F(c, c^\dagger)]] = \mathcal{N}_q\left[c \sum_{\gamma\in\mathscr{F}(F(c,c^\dagger))} \mathscr{W}_q(\gamma) : \gamma :\right],$$

which is equivalent to (write $: \gamma : = (c^\dagger)^{a_\gamma} c^{b_\gamma}$)

$$\mathcal{N}_q[cF(c, c^\dagger)] = \sum_{\gamma\in\mathscr{F}(F(c,c^\dagger))} \mathscr{W}_q(\gamma)\mathcal{N}_q[c(c^\dagger)^{a_\gamma} c^{b_\gamma}].$$

Using $\mathcal{N}_q[c(c^\dagger)^{a_\gamma}c^{b_\gamma}] = q^{a_\gamma}(c^\dagger)^{a_\gamma}c^{b_\gamma+1} + [a_\gamma]_q(c^\dagger)^{a_\gamma-1}c^{b_\gamma}$, we obtain

$$\mathcal{N}_q[cF(c,c^\dagger)] = \sum_{\gamma\in\mathscr{F}(F(c,c^\dagger))} \mathscr{W}_q(\gamma)\{q^{a_\gamma}(c^\dagger)^{a_\gamma}c^{b_\gamma+1} + [a_\gamma]_q(c^\dagger)^{a_\gamma-1}c^{b_\gamma}\}. \tag{8}$$

This is the explicit expression for the left-hand side of (7) in the present case. We will now study the right-hand side of (7) in the present case and show that it yields the same result as (8), thus proving the assertion for case (II). As the first step, we write $\mathscr{F}(cF(c,c^\dagger)) = \mathscr{F}^+(cF(c,c^\dagger)) \cup \mathscr{F}^-(cF(c,c^\dagger))$ where $\mathscr{F}^+(cF(c,c^\dagger))$ denotes the subset of Feynman diagrams where an edge starts at the most left vertex c and where $\mathscr{F}^-(cF(c,c^\dagger))$ denotes the set of the remaining Feynman diagrams where the most left vertex c is a singleton. Thus,

$$\sum_{\gamma\in\mathscr{F}(cF(c,c^\dagger))} \mathscr{W}_q(\gamma) : \gamma : = \sum_{\delta\in\mathscr{F}^-(cF(c,c^\dagger))} \mathscr{W}_q(\delta) : \delta : + \sum_{\beta\in\mathscr{F}^+(cF(c,c^\dagger))} \mathscr{W}_q(\beta) : \beta : . \tag{9}$$

Note that there is a bijection between $\mathscr{F}^-(cF(c,c^\dagger))$ and $\mathscr{F}(F(c,c^\dagger))$ associating with $\delta \in \mathscr{F}^-(cF(c,c^\dagger))$ the Feynman diagram $\delta' \in \mathscr{F}(F(c,c^\dagger))$ by deleting the most left vertex c in the word $cF(c,c^\dagger)$. However, the q-weights of the two Feynman diagrams δ and δ' are not equal. The total crossing numbers are equal, i.e. $tc(\delta) = tc(\delta')$, whereas the lengths are related by $l(\delta) = l(\delta') + a_{\delta'}$. Using 6, this yields for the q-weights $\mathscr{W}_q(\delta) = q^{a_{\delta'}}\mathscr{W}_q(\delta')$, implying $\mathscr{W}_q(\delta) : \delta : = q^{a_{\delta'}}\mathscr{W}_q(\delta') : \delta' : c$. Thus, the first sum in (9) can be written as

$$\sum_{\delta\in\mathscr{F}^-(cF(c,c^\dagger))} \mathscr{W}_q(\delta) : \delta : = \sum_{\gamma\in\mathscr{F}(F(c,c^\dagger))} \mathscr{W}_q(\gamma) q^{a_\gamma}(c^\dagger)^{a_\gamma}c^{b_\gamma+1}, \tag{10}$$

where we have switched to the more convenient notation $\delta' \rightsquigarrow \gamma$. Let us turn to the second sum in (9). In analogy to an argument made above, we introduce the map

$$\mathscr{R} : \mathscr{F}^+(cF(c,c^\dagger)) \to \mathscr{F}(F(c,c^\dagger)),$$

which (i) deletes the edge beginning at the most left vertex c in the word $cF(c,c^\dagger)$ and (ii) deletes the most left vertex c. Clearly, this map is well defined. In contrast to above this is not a bijection, since there can be many $\beta \in \mathscr{F}^+(cF(c,c^\dagger))$ which are mapped onto the same $\beta' \in \mathscr{F}(F(c,c^\dagger))$. Let us denote the preimage of β' under this map by $\mathscr{F}^+_{\beta'}(cF(c,c^\dagger))$, i.e.

$$\mathscr{F}^+_{\beta'}(cF(c,c^\dagger)) = \{\beta \in \mathscr{F}^+(cF(c,c^\dagger)) | \mathscr{R}(\beta) = \beta' \in \mathscr{F}(F(c,c^\dagger))\}.$$

Since these preimages are disjoint, we can write

$$\begin{aligned}\sum_{\beta\in\mathscr{F}^+(cF(c,c^\dagger))} \mathscr{W}_q(\beta) : \beta : &= \sum_{\beta'\in\mathscr{F}(F(c,c^\dagger))} \sum_{\beta\in\mathscr{F}^+_{\beta'}(cF(c,c^\dagger))} \mathscr{W}_q(\beta) : \beta : \\ &= \sum_{\beta'\in\mathscr{F}(F(c,c^\dagger))} \left\{ \sum_{\beta\in\mathscr{F}^+_{\beta'}(cF(c,c^\dagger))} \mathscr{W}_q(\beta) \right\} (c^\dagger)^{-1} : \beta' :,\end{aligned}$$

where we have used the fact that for all $\beta \in \mathscr{F}^+_{\beta'}(cF(c,c^\dagger))$ one has $: \beta : = (c^\dagger)^{-1} : \beta' :$. Note that this formal notation is not meaningless in indicating that the degree of the creation operator has to be decreased by 1, since, by definition, there is at least one singleton of type $\mathscr{C}$ in every β', namely the one which becomes 'free' after deleting the edge in step (i) from above. Let us write as above $: \beta' : = (c^\dagger)^{a_{\beta'}}c^{b_{\beta'}}$. Assuming for the moment

$$\sum_{\beta\in\mathscr{F}^+_{\beta'}(cF(c,c^\dagger))} \mathscr{W}_q(\beta) = [a_{\beta'}]_q \mathscr{W}_q(\beta'), \tag{11}$$

we have, therefore, shown that

$$\sum_{\beta \in \mathscr{F}^{+}(cF(c,c^{\dagger}))} \mathscr{W}_q(\beta) : \beta : \; = \sum_{\beta' \in \mathscr{F}(F(c,c^{\dagger}))} \mathscr{W}_q(\beta')[a_{\beta'}]_q (c^{\dagger})^{a_{\beta'}-1} c^{b_{\beta'}}. \tag{12}$$

Switching to the more convenient notation $\beta' \rightsquigarrow \gamma$ and inserting (10) and (12) into the right-hand side of (9) yields

$$\sum_{\gamma \in \mathscr{F}(cF(c,c^{\dagger}))} \mathscr{W}_q(\gamma) : \gamma : \; = \sum_{\gamma \in \mathscr{F}(F(c,c^{\dagger}))} \mathscr{W}_q(\gamma)\{q^{a_\gamma}(c^{\dagger})^{a_\gamma} c^{b_\gamma+1} + [a_\gamma]_q (c^{\dagger})^{a_\gamma - 1} c^{b_\gamma}\}. \tag{13}$$

Comparing (8) and (13) shows that $\mathcal{N}_q[cF(c,c^{\dagger})] = \sum_{\gamma \in \mathscr{F}(cF(c,c^{\dagger}))} \mathscr{W}_q(\gamma) : \gamma :$, provided (11) holds true. We are now going to show (11). First note that $\left|\mathscr{F}^{+}_{\beta'}(cF(c,c^{\dagger}))\right| = a_{\beta'}$. The Feynman diagrams $\beta_i \in \mathscr{F}^{+}_{\beta'}(cF(c,c^{\dagger}))$ with $1 \leqslant i \leqslant a_{\beta'}$ are easy to describe: β_i arises from β' by adjoining the most left vertex c to the word $F(c,c^{\dagger})$ and connecting it via an edge with the ith singleton of type $\mathscr{C}$ in the word $cF(c,c^{\dagger})$. We are now going to show that

$$\mathscr{W}_q(\beta_i) = q^{i-1}\mathscr{W}_q(\beta'). \tag{14}$$

This implies

$$\sum_{\beta \in \mathscr{F}^{+}_{\beta'}(cF(c,c^{\dagger}))} \mathscr{W}_q(\beta) = \sum_{i=1}^{a_{\beta'}} \mathscr{W}_q(\beta_i) = \sum_{i=1}^{a_{\beta'}} q^{i-1}\mathscr{W}_q(\beta') = [a_{\beta'}]_q \mathscr{W}_q(\beta'),$$

i.e. the sought-for equality (11). To show (14), we depict the word $cF(c,c^{\dagger})$ in the following fashion:

$$cF(c,c^{\dagger}) \equiv v_0 R_1 v_1 R_2 v_2 \cdots R_{a_{\beta'}} v_{a_{\beta'}} R_{a_{\beta'}+1},$$

where v_0 denotes the most left vertex c, the vertices v_i with $1 \leqslant i \leqslant a_{\beta'}$ denote the singletons of type $\mathscr{C}$ and (possibly empty) R_i the 'blocks' in between. Let us denote by α_i the number of singletons of type $\mathscr{A}$ in R_i. For the Feynman diagram β', we denote by $\nu_{i,k}$ the number of edges starting in R_i and ending behind the vertex v_k. Let us now consider the Feynman diagrams β_i and start with the case $i = 1$. Due to the additional edge between v_0 and v_1, one has the following relations:

$$c(\beta_1) = c(\beta') + \nu_{1,1}, \qquad d(\beta_1) = d(\beta') + \alpha_1 - \nu_{1,1}, \qquad l(\beta_1) = l(\beta') - \alpha_1,$$

showing that $tc(\beta_1) + l(\beta_1) = tc(\beta') + l(\beta')$ and, therefore, $\mathscr{W}_q(\beta_1) = \mathscr{W}_q(\beta')$. Let us consider the case $i = 2$. Due to the additional edge between v_0 and v_2 there will be $\nu_{1,2} + \nu_{2,2}$ additional crossings in β_2 compared to β', i.e. $c(\beta_2) = c(\beta') + \nu_{1,2} + \nu_{2,2}$. Turning to the degenerate crossings, there are—in comparison to β'—three effects which have to be considered.

(1) Since the vertex v_2 is no more a singleton there will be $\nu_{1,2} + \nu_{2,2}$ degenerate crossings less.
(2) Since there is a new edge between v_0 and v_2 there will be $\alpha_1 + \alpha_2$ degenerate crossings more, coming from the covered blocks R_1 and R_2.
(3) Since v_1 lies in between v_0 and v_2, the triple $v_0 < v_1 < v_2$ also accounts for an additional degenerate crossing.

This shows that $d(\beta_2) = d(\beta') + (\alpha_1 + \alpha_2) - (\nu_{1,2} + \nu_{2,2}) + 1$. The length of β_2 results by decreasing the length of β' by $\alpha_1 + \alpha_2$ since the vertex v_2 is no more a singleton. Collecting the above results shows that $tc(\beta_2) + l(\beta_2) = tc(\beta') + l(\beta') + 1$, implying $\mathscr{W}_q(\beta_2) = q\mathscr{W}_q(\beta')$. Let us now turn to the case of general i (with $1 \leqslant i \leqslant a_{\beta'}$) where one has the analogous 'trade off' between the different parts which go into the q-weight (leaving, in effect, only the vertices $v_1, \ldots, v_{i-1}$ as contributors to the difference between β_i and β'). The same argument as above shows that

$$c(\beta_i) = c(\beta') + \sum_{k=1}^{i} \nu_{k,i}, \qquad d(\beta_i) = d(\beta') + \sum_{k=1}^{i} \alpha_k - \sum_{k=1}^{i} \nu_{k,i} + (i-1),$$
$$l(\beta_i) = l(\beta') + \sum_{k=1}^{i} \alpha_k.$$

It follows that $tc(\beta_i) + l(\beta_i) = tc(\beta') + l(\beta') + (i-1)$ and, therefore, that $\mathscr{W}_q(\beta_i) = q^{i-1}\mathscr{W}_q(\beta')$. But this is exactly (14) which was to be shown. Thus, the proof for case (II) is complete. □

Clearly, letting $q = 1$ reduces (7) to the undeformed case (5). Note that very similar results have been derived in [25–27] with a slightly different point of view. In [12] and [29], different graphical means for normal ordering bosonic operators are discussed.

4. Examples and consequences

Let us consider a simple example for theorem 3.1 before we draw some connections to q-rook numbers and Stirling numbers.

Example 4.1. Let $F(c, c^\dagger) = c^2c^\dagger c^2c^\dagger = ccc^\dagger ccc^\dagger$. Since there are only two creators in the word, the Feynman diagrams can have degree at most 2. The trivial Feynman diagram γ of degree 0 yields : γ : $= (c^\dagger)^2c^4$ and has q-weight $\mathscr{W}_q(\gamma) = q^{l(\gamma)} = q^{2+2+0+1+1+0}$. Thus, the Feynman diagram of degree 0 yields the contribution $q^6(c^\dagger)^2c^4$. There are six Feynman diagrams of degree 1, namely

$$\{(1,3), (1,6), (2,3), (2,6), (4,6), (5,6)\},$$

and their q-weights are given by (same order) $\{q^4, q^5, q^3, q^4, q^3, q^2\}$. Thus, the Feynman diagrams of degree 1 yield the contribution $(q^2 + 2q^3 + 2q^4 + q^5)c^\dagger c^3$. There are six Feynman diagrams of degree 2, namely

$$\{(1,3)(2,6), (1,3)(4,6), (1,3)(5,6), (1,6)(2,3), (2,3)(4,6), (2,3)(5,6)\}$$

with q-weights $\{q^3, q^2, q, q^2, q, 1\}$, yielding the contribution $(1 + 2q + 2q^2 + q^3)c^2$. Thus,

$$\mathcal{N}_q[c^2c^\dagger c^2c^\dagger] = q^6(c^\dagger)^2c^4 + (q^2 + 2q^3 + 2q^4 + q^5)c^\dagger c^3 + (1 + 2q + 2q^2 + q^3)c^2, \tag{15}$$

which may also be calculated by hand from (1).

We now want to draw a connection between theorem 3.1 and the results of Varvak [14]. Given a word $w = F(c, c^\dagger)$ containing m creation operators $c^\dagger$ and n annihilation operators c (with $n \leqslant m$), she associates with w a certain Ferrers board B_w outlined by w. Denoting by $R_k(B_w, q)$ the kth q-rook number of the board B_w, she shows that ([14], theorem 6.1)

$$w = \sum_{k=0}^{n} R_k(B_w, q)(c^\dagger)^{m-k}c^{n-k}.$$

Using theorem 3.1, we first note that the set of Feynman diagrams is the disjoint union of Feynman diagrams of degree k, i.e. $\mathscr{F}(w) = \cup_{k=0}^{n}\mathscr{F}_k(w)$, and that $\gamma \in \mathscr{F}_k(w)$ implies

$: \gamma : = (c^\dagger)^{m-k} c^{n-k}$, yielding

$$w = \sum_{k=0}^{n} \left\{ \sum_{\gamma \in \mathscr{F}_k(w)} \mathscr{W}_q(\gamma) \right\} (c^\dagger)^{m-k} c^{n-k}.$$

Comparing the two expressions yields the following corollary.

Corollary 4.2. *Given a word $w = F(c, c^\dagger)$, the kth q-rook number of the associated Ferrers board B_w equals the sum of the q-weights of all Feynman diagrams of degree k on w, i.e.*

$$R_k(B_w, q) = \sum_{\gamma \in \mathscr{F}_k(w)} \mathscr{W}_q(\gamma). \tag{16}$$

Let us return to the undeformed case ($q = 1$) and consider the particular word $F(c, c^\dagger) = (c^\dagger c)^n$. The same argument as above shows that

$$\mathcal{N}[(c^\dagger c)^n] = \sum_{k=0}^{n} \sum_{\gamma \in \mathscr{F}_{n-k}((c^\dagger c)^n)} (c^\dagger)^k c^k.$$

Comparing this with the undeformed case of (3) shows that the conventional Stirling numbers of the second kind can also be interpreted as the number of Feynman diagrams of degree $n-k$ on the particular word $c^\dagger c c^\dagger c \cdots c^\dagger c$ of length $2n$, i.e.

$$S(n, k) = |\mathscr{F}_{n-k}((c^\dagger c)^n)|. \tag{17}$$

Turning to the q-deformed situation, the same argument shows that

$$S_q(n, k) = \sum_{\gamma \in \mathscr{F}_{n-k}((c^\dagger c)^n)} \mathscr{W}_q(\gamma) = \sum_{\gamma \in \mathscr{F}_{n-k}((c^\dagger c)^n)} q^{tc(\gamma)+l(\gamma)}. \tag{18}$$

Remark 4.3. In [18] the study of combinatorial aspects of the normal ordering of multi-mode boson operators was begun, and interesting combinatorial questions were addressed. In view of the above q-Wick's theorem, it is natural to consider the analogous problem for the q-deformed variant of the multi-mode boson operator. However, this does not seem to be easy as the following example will show. For concreteness, we consider the deformed two-mode boson having the commutation relations

$$[a, a^\dagger]_{q_a} = 1, \qquad [b, b^\dagger]_{q_b} = 1, \qquad [a, b^\dagger]_{q_{ab}} = 1$$

and all other commutators vanish (note, in particular, that the two modes interact nontrivially!). Here, the deformation parameters q_a, q_b, q_{ab} are in the moment arbitrary. Let us consider the simple example $F(a, a^\dagger, b, b^\dagger) = abb^\dagger$. First commuting b and $b^\dagger$ using the above commutation relations yields $abb^\dagger = q_b ab^\dagger b + a$; then commuting a and $b^\dagger$ yields $abb^\dagger = q_b q_{ab} b^\dagger ab + q_b b + a$. On the other hand, first commuting a and b and then commuting $b^\dagger$ to the left yields $abb^\dagger = q_b q_{ab} b^\dagger ba + q_{ab} a + b$. Clearly, for the two results to coincide we have to assume that $q_b = q_{ab} = 1$. A similar computation for $baa^\dagger$ also shows that $q_a = 1$. Thus, it seems that a q-deformed version of the multi-mode boson considered in [18] does not exist.

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